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New State Transition Matrix for Relative Motion on an Arbitrary Elliptical Orbit

Koji Yamanaka
National Space Development Agency, Tokyo 105-8060, Japan
and
Finn Ankersen
European Space Research and Technology Center, 2200 AG Noordwijk, The Netherlands

A new state transition matrix is described for the nonlinear problem of relative motion on an arbitrary elliptical orbit. A linearization is performed, leading to a set of linear differential equations with time-dependent coefficients. A new and simpler solution to those equations is represented in a convenient state transition matrix form. This new state transition matrix is valid for arbitrary elliptical orbits of $0 \leq e < 1$. The state propagation using the new state transition matrix shows good agreement with numerical results.

Nomenclature

$\boldsymbol{a}_{cd}$ = acceleration due to forces other than the inverse square gravity term on the chaser spacecraft, for example, solar pressure, air drag, higher gravity terms, etc.
$\boldsymbol{a}_f$ = acceleration due to thrust forces on the chaser spacecraft
$\boldsymbol{a}_{td}$ = acceleration due to forces other than the inverse square gravity term on the target spacecraft, for example, solar pressure, air drag, higher gravity terms, etc.
c = $\rho \cos\theta$
e = eccentricity of target orbit
h = orbital angular momentum of target orbit
k = $\sqrt{(h/p^2)}$
$\boldsymbol{R}$ = vector from the center of gravity to the target spacecraft
$\boldsymbol{r}$ = vector from the target spacecraft to the chaser spacecraft
s = $\rho \sin\theta$
T = transpose of vector or matrix
t = time
x, y, z = chaser relative state vector in the target orbital coordinate frame
θ = true anomaly
μ = gravity constant
ρ = $1 + e\cos\theta$
ω = orbital rate of target

Introduction

THE Clohessy–Wiltshire (CW) equations of motion describe the proximity relative motion for circular orbits. The CW equations are valid if two conditions are satisfied, namely, that the distance between the chaser and the target is small compared with the distance between the target and the center of the Earth and that the target orbit is near circular. The CW equations consist of three simple differential equations, which can be solved analytically. These solutions have been widely used for circular orbit rendezvous in practice. On the other hand, for future rendezvous (RV) missions, such as servicing vehicles or sample returns from small celestial

Koji Yamanaka was born in 1968 in Mie Prefecture, Japan. He received his M.S. in aeronautical engineering in 1992 from Nagoya University. He has been working for the National Space Development Agency of Japan since then. He has worked on the automated rendezvous experiment of Engineering Test Satellite VII, Japanese lunar lander, and currently on the H-II Transfer Vehicle project, which is a space station logistic support vehicle. His research interests are in the area of automated rendezvous and formation flying. He is a member of AIAA.

Finn Ankersen was born in 1958 in Aalborg, Denmark. He received his M.Sc. in electrical engineering in 1985 from Aalborg University, with a specialization in automatic control systems there and at the Massachusetts Institute of Technology. He worked for a year in the software industry before joining the European Space Agency Research and Technology Center in the Netherlands, where he holds a senior position in the spacecraft guidance, navigation, and control technical section. His research interests are in the area of multivariable robust control, system identification, mathematical modeling and design, and simulation tools design for space systems and electrooptical mechanical systems.

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objects, the target orbits are not necessarily circular. It is well known that the CW equations, or equivalently the analytical CW solutions, cause significant error if they are applied to fairly elliptical orbits.

The possibility of propagating an initial state to a final state without numerical integration is particularly important for RV problems. Many attempts to solve the differential equations of relative motion for the elliptical orbit of arbitrary eccentricity can be seen at several places in the literature. Carter summarized them in 1998 (Ref. 1). However, a simple and well-arranged transition matrix applicable to arbitrary elliptic orbits seems not to have been obtained so far. The state transition matrices of Carter[1] or Wolfsberger et al.,[2] as examples, are not particularly simple for engineering use. Melton[3] showed another solution method, which is valid for in-plane and out-of-plane motion, but is approximate and loses accuracy with higher values of eccentricity. In this paper a new and simpler solution to differential equations of relative motion on an arbitrary elliptical orbit is developed and described in a convenient state transition matrix form. This new state transition matrix has no singularity at $e = 0$ and is valid for an arbitrary elliptical orbit $0 \leq e < 1$.

Differential Equations of Proximity Relative Motion

Differential equations of proximity relative motion can be found in many places in the literature, but are described here briefly for the reader's convenience. In Fig. 1, $\boldsymbol{R}$ is the vector from the center of gravity to the target spacecraft and $\boldsymbol{r}$ is the vector from the target spacecraft to the chaser spacecraft. The equation of motion for the target spacecraft in the inertial frame is

$$\ddot{\boldsymbol{R}} = -\mu(\boldsymbol{R}/|\boldsymbol{R}|^3) + \boldsymbol{a}_{td} \tag{1}$$

The equation of motion of the chaser spacecraft is written as

$$\ddot{\boldsymbol{R}} + \ddot{\boldsymbol{r}} = -\mu[(\boldsymbol{R}+\boldsymbol{r})/(|\boldsymbol{R}+\boldsymbol{r}|^3)] + \boldsymbol{a}_f + \boldsymbol{a}_{cd} \tag{2}$$

Rewriting part of the chaser equation we obtain

$$|\boldsymbol{R}+\boldsymbol{r}| = [(\boldsymbol{R}+\boldsymbol{r})^T(\boldsymbol{R}+\boldsymbol{r})]^{\frac{1}{2}} = (|\boldsymbol{R}|^2 + 2\boldsymbol{R}^T\boldsymbol{r} + |\boldsymbol{r}|^2)^{\frac{1}{2}}$$

$$\therefore |\boldsymbol{R}+\boldsymbol{r}|^3 = |\boldsymbol{R}|^3[1 + 2(\boldsymbol{R}^T\boldsymbol{r}/|\boldsymbol{R}|^2) + |\boldsymbol{r}|^2/|\boldsymbol{R}|^2]^{\frac{3}{2}} \tag{3}$$

If the distance between the chaser and the target is much smaller than the distance between the target and center of the gravity field, that is, $|\boldsymbol{R}| \gg |\boldsymbol{r}|$, then

$$\frac{\boldsymbol{R}+\boldsymbol{r}}{|\boldsymbol{R}+\boldsymbol{r}|^3} = \frac{\boldsymbol{R}+\boldsymbol{r}}{|\boldsymbol{R}|^3}\left(1 + 2\frac{\boldsymbol{R}^T\boldsymbol{r}}{|\boldsymbol{R}|^2} + \frac{|\boldsymbol{r}|^2}{|\boldsymbol{R}|^2}\right)^{-\frac{3}{2}} \approx \frac{\boldsymbol{R}+\boldsymbol{r}}{|\boldsymbol{R}|^3}$$
$$\times\left[1 - \frac{3}{2}\left(2\frac{\boldsymbol{R}^T\boldsymbol{r}}{|\boldsymbol{R}|^2} + \frac{|\boldsymbol{r}|^2}{|\boldsymbol{R}|^2}\right)\right] \approx \frac{1}{|\boldsymbol{R}|^3}\left(\boldsymbol{R}+\boldsymbol{r} - 3\frac{\boldsymbol{R}^T\boldsymbol{r}}{|\boldsymbol{R}|^2}\boldsymbol{R}\right) \tag{4}$$

This approximation is valid for terminal RV because, for example, for the Earth, $|\boldsymbol{R}|$ is 6878 km for 500-km altitude target orbit and $|\boldsymbol{r}|$ is less than 10 km for most terminal RV. Note that this proximity requirement can be relaxed significantly if one uses cylindrical coordinates.

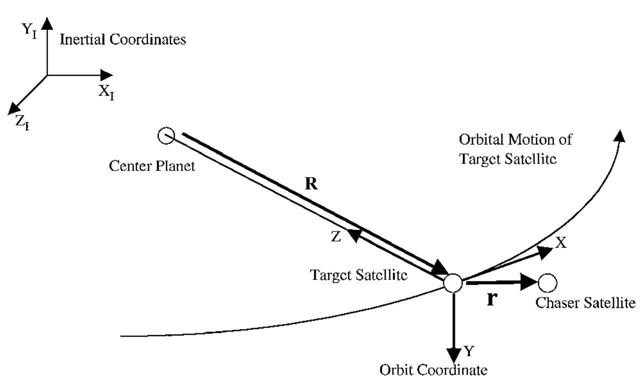

Fig. 1 Coordinates and definition.

When Eq. (1) is subtracted from Eq. (2) using this approximation, the equation of relative motion can be obtained in the inertial frame as

$$\ddot{\boldsymbol{r}} = -(\mu/|\boldsymbol{R}|^3)[\boldsymbol{r} - 3(\boldsymbol{R}^T\boldsymbol{r}/|\boldsymbol{R}|^2)\boldsymbol{R}] + \boldsymbol{a}_f + \boldsymbol{a}_{cd} - \boldsymbol{a}_{td} \tag{5}$$

It is convenient to describe Eq. (5) in a target-orbital coordinate system (see Fig. 1), where the origin of the coordinate system is the position of the target and the Z axis points in the nadir direction. The Y axis is normal to the orbital plane, opposite the angular momentum vector, and the X axis completes the right-hand system. This orbital coordinate system is usually used for describing the relative dynamics between two spacecraft. In general, the time derivative of an arbitrary vector $\boldsymbol{A}$ in an inertial coordinate system can be expressed in terms of the deviation with respect to a rotating coordinate system as

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \left.\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right|_{\mathrm{rot}} + \omega \times \boldsymbol{A} \tag{6}$$

When Eq. (6) is applied twice, Eq. (5) can now be written in target-orbital coordinates as

$$\ddot{\boldsymbol{r}} = -(\mu/|\boldsymbol{R}|^3)[\boldsymbol{r} - 3(\boldsymbol{R}^T\boldsymbol{r}/|\boldsymbol{R}|^2)\boldsymbol{R}] - 2(\omega \times \dot{\boldsymbol{r}}) - \dot{\omega} \times \boldsymbol{r}$$
$$- \omega \times (\omega \times \boldsymbol{r}) + \boldsymbol{a}_f + \boldsymbol{a}_{cd} - \boldsymbol{a}_{td} \tag{7}$$

The dot and double dot indicates the time derivative with respect to the rotating coordinate system.

The components of each vector in the target orbit coordinate system are

$$\omega = \begin{bmatrix} 0 \\ -\omega \\ 0 \end{bmatrix}, \qquad \boldsymbol{R} = \begin{bmatrix} 0 \\ 0 \\ -R \end{bmatrix} \tag{8}$$

where ω is the target-orbital rate and R is the distance from the center of the Earth to the target spacecraft. The chaser relative state vector in the orbital frame is

$$\boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9}$$

The terms in Eq. (7) become

$$\omega \times \dot{\boldsymbol{r}} = \begin{bmatrix} -\omega\dot{z} \\ 0 \\ \omega\dot{x} \end{bmatrix} \tag{10}$$

$$\dot{\omega} \times \boldsymbol{r} = \begin{bmatrix} -\dot{\omega}z \\ 0 \\ \dot{\omega}x \end{bmatrix} \tag{11}$$

$$\omega \times (\omega \times \boldsymbol{r}) = \begin{bmatrix} -\omega^2 x \\ 0 \\ -\omega^2 z \end{bmatrix} \tag{12}$$

$$\boldsymbol{r} - 3\frac{\boldsymbol{R}^T\boldsymbol{r}}{|\boldsymbol{R}|^2}\boldsymbol{R} = \begin{bmatrix} x \\ y \\ -2z \end{bmatrix} \tag{13}$$

From the relation

$$R^2\omega = h$$

where h is the orbital angular momentum of target, we can define the constant k as

$$\mu/R^3 = \left(\mu/h^{\frac{3}{2}}\right)\omega^{\frac{3}{2}} \equiv k\omega^{\frac{3}{2}}, \qquad k \equiv \mu/h^{\frac{3}{2}} = \mathrm{const} \tag{14}$$

When these relations are substituted into Eq. (7), the equations of proximity relative motion in the target-orbital coordinate system become

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}-k\omega^{\frac{3}{2}}x + 2\omega\dot{z} + \dot{\omega}z + \omega^2 x\\ -k\omega^{\frac{3}{2}}y\\ 2k\omega^{\frac{3}{2}}z - 2\omega\dot{x} - \dot{\omega}x + \omega^2 z\end{bmatrix} + \boldsymbol{a}_f + \boldsymbol{a}_{cd} - \boldsymbol{a}_{td} \tag{15}$$

The only assumption made is that the distance between the chaser and the target is small compared to the distance between the target and the center of the Earth. Equation (15) is, therefore, applicable to orbits of arbitrary eccentricity.

In the next section, we will find the homogeneous solution to Eq. (15):

$$\boldsymbol{a}_f = 0, \qquad \boldsymbol{a}_{cd} = \boldsymbol{a}_{td} \tag{16}$$

This means the chaser is flying in free motion, and the external forces on the chaser and the target are identical. Our final goal is to obtain a simple state transition matrix for RV engineering.

Simplification of the Equations of Motion

Among the methods to simplify Eq. (15), the way that is shown in Ref. 1 seems to obtain the simplest final form. This method adopts the true anomaly θ of the target spacecraft as an independent variable instead of time t and also adopts the transformation

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{bmatrix} = (1 + e\cos\theta)\begin{bmatrix}x\\ y\\ z\end{bmatrix} \tag{17}$$

Because the true anomaly θ is a monotonically increasing variable with time t, the independent variable can be changed from time t to the true anomaly θ of the target spacecraft. The derivative of an arbitrary variable a with respect to time t then becomes

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}a}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}a}{\mathrm{d}\theta} \tag{18}$$

and furthermore,

$$\begin{aligned}\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right) = \omega\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\omega\frac{\mathrm{d}a}{\mathrm{d}\theta}\right)\\ &= \omega\left(\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}a}{\mathrm{d}\theta} + \omega\frac{\mathrm{d}^2 a}{\mathrm{d}\theta^2}\right) = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}a}{\mathrm{d}\theta} + \omega^2\frac{\mathrm{d}^2 a}{\mathrm{d}\theta^2}\end{aligned} \tag{19}$$

Writing the derivative with respect to true anomaly θ as

$$\frac{\mathrm{d}a}{\mathrm{d}\theta} = a' \tag{20}$$

Eqs. (18) and (19) can be expressed as

$$\dot{a} = \omega a', \qquad \ddot{a} = \omega^2 a'' + \omega\omega' a' \tag{21}$$

Changing the independent variable, the components of Eq. (15) can be expressed with respect to true anomaly θ as

$$\omega^2 x'' + \omega\omega' x' = \left(\omega^2 - k\omega^{\frac{3}{2}}\right)x + 2\omega^2 z' + \omega\omega' z \tag{22}$$

$$\omega^2 y'' + \omega\omega' y' = -k\omega^{\frac{3}{2}} y \tag{23}$$

$$\omega^2 z'' + \omega\omega' z' = \left(\omega^2 + 2k\omega^{\frac{3}{2}}\right)z - 2\omega^2 x' - \omega\omega' x \tag{24}$$

From the angular momentum, we obtain

$$\omega = (h/R^2) = (h/p^2)(1 + e\cos\theta)^2 = k^2(1 + e\cos\theta)^2 = k^2\rho^2$$

$$\rho \equiv 1 + e\cos\theta \tag{25}$$

$$\omega' = 2k^2\rho\rho' = -2k^2 e\sin\theta\rho \tag{26}$$

Therefore, the components of the equations of relative motion become

$$\rho x'' - 2e\sin\theta x' - e\cos\theta x = 2\rho z' - 2e\sin\theta z \tag{27}$$

$$\rho y'' - 2e\sin\theta y' = -y \tag{28}$$

$$\rho z'' - 2e\sin\theta z' - (3 + e\cos\theta)z = -2\rho x' + 2e\sin\theta x \tag{29}$$

Changing the dependent variables as follows:

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{bmatrix} = \rho\begin{bmatrix}x\\ y\\ z\end{bmatrix} \tag{30}$$

we obtain, for example,

$$\tilde{x}' = \rho x' - e\sin\theta x \tag{31}$$

$$\tilde{x}'' = \rho x'' - 2e\sin\theta x' - e\cos\theta x \tag{32}$$

and similarly for the other axes. Consequently, the equations of relative motion become rather simple:

$$\tilde{x}'' = 2\tilde{z}' \tag{33}$$

$$\tilde{y}'' = -\tilde{y} \tag{34}$$

$$\tilde{z}'' = 3\tilde{z}/\rho - 2\tilde{x}' \tag{35}$$

The y component is the equation of the harmonic oscillator. It can be analytically integrated as

$$\tilde{y} = K_{y1}\sin\theta + K_{y2}\cos\theta \tag{36}$$

As for x, z, integrating Eq. (33) once, it becomes

$$\tilde{x}' = 2\tilde{z} + K_{x1} \tag{37}$$

where K_{x1} is a constant of integration.

When this equation is inserted into Eq. (35), it becomes

$$\tilde{z}'' + (4 - 3/\rho)\tilde{z} = -2K_{x1} \tag{38}$$

The problem is now reduced to solving the preceding differential equation for $\tilde{z}$.

Proposal of New Analytical Solution

One of the solutions of the homogeneous second-order differential equation

$$\tilde{z}'' + (4 - 3/\rho)\tilde{z} = 0 \tag{39}$$

is

$$\varphi_1 = \rho\sin\theta \tag{40}$$

According to Carter,[1] another solution is

$$\varphi_2 = \rho\sin\theta\int_{\theta_0}^{\theta}\frac{1}{\sin^2\tau\rho(\tau)^2}\,\mathrm{d}\tau \tag{41}$$

As Carter[1] mentioned, φ_2 becomes singular at $\theta = \pm\pi$. Carter[4] proposed another solution of the form

$$\varphi_2 = 2e\rho\sin\theta\int_{\theta_0}^{\theta}\frac{\cos\tau}{\rho(\tau)^3}\,\mathrm{d}\tau - \frac{\cos\theta}{\rho} \tag{42}$$

which can be obtained by partial integration of Eq. (41). In this solution, the singularity at $\theta = \pm\pi$ has been removed, and the solution becomes valid for all true anomalies. However, the state transition matrix becomes singular at $e = 0$. To avoid this singularity Carter[1] transformed the integrand in Eq. (42) into

$$\int_{\theta_0}^{\theta}\frac{\cos\tau}{\rho(\tau)^3}\,\mathrm{d}\tau = \frac{\sin\theta}{\rho^3} - 3e\int_{\theta_0}^{\theta}\frac{\sin^2\tau}{\rho(\tau)^4}\,\mathrm{d}\tau \tag{43}$$

The state transition matrix no longer has a singularity at $e = 0$. However, as a consequence, the state transition matrix is not simple for engineering use.

To overcome these problems, we propose to use a new integral term instead of the integral term that can be seen in Eq. (41) or (42):

$$J(\theta) = \int_{\theta_0}^{\theta} \frac{1}{\rho(\tau)^2}\,\mathrm{d}\tau \tag{44}$$

The motivation for this form is the constant of angular momentum given in Eq. (25)

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = k^2\rho^2 \tag{45}$$

so that

$$k^2\,\mathrm{d}t = (1/\rho^2)\,\mathrm{d}\theta \tag{46}$$

Integrating both sides yields

$$k^2(t - t_0) = \int_{\theta_0}^{\theta} \frac{1}{\rho(\tau)^2}\,\mathrm{d}\tau = J(\theta) \tag{47}$$

This means that the integral term J can be calculated directly from the transition time, and therefore, there is no need to evaluate the integral directly.

To find the solution using this integration, we assume the solution is of the form

$$\varphi_2 = c_1\rho\sin\theta J + \rho\cos\theta + c_2 \tag{48}$$

where c_1 and c_2 are constants. To simplify the notation, we substitute the following variables:

$$\phi \equiv \rho\sin\theta, \qquad \xi \equiv \rho\cos\theta \tag{49}$$

Inserting

$$\varphi_2 = c_1\phi J + \xi + c_2 \tag{50}$$

into

$$\tilde{z}'' + (4 - 3/\rho)\tilde{z} = 0 \tag{51}$$

we get the following identity:

$$c_1\{[\phi'' + (4 - 3/\rho)\phi]J + 2\phi' J' + \phi J''\} + \xi'' + (4 - 3/\rho)\xi + c_2(4 - 3/\rho) = 0 \tag{52}$$

When the two equations

$$\phi'' + (4 - 3/\rho)\phi = 0, \qquad \xi'' + (4 - 3/\rho)\xi = 2e \tag{53}$$

are inserted, this identity becomes simply

$$c_1(2\phi' J' + \phi J'') + 2e + c_2(4 - 3/\rho) = 0 \tag{54}$$

The first term in parenthesis is identified as

$$2\phi' J' + \phi J'' = (2/\rho)\cos\theta \tag{55}$$

hence,

$$(2c_1/\rho)\cos\theta + 2e + c_2(4 - 3/\rho) = 0 \tag{56}$$

Rearranging it in terms of $\cos\theta$,

$$\left(2c_1 + 4ec_2 + 2e^2\right)\cos\theta + c_2 + 2e = 0$$

$$\therefore c_1 = 3e^2, \qquad c_2 = -2e \tag{57}$$

Finally, we get a new solution:

$$\varphi_2 = 3e^2\rho\sin\theta J + \rho\cos\theta - 2e \tag{58}$$

This is the solution we propose in this paper. The benefit of this is that the solutions of Eqs. (33–35) can be written in a very simple form.

The linear independence of the two solutions needs to be checked:

$$\varphi_1 = \rho\sin\theta \tag{59}$$

$$\varphi_2 = 3e^2\rho\sin\theta J + \rho\cos\theta - 2e \tag{60}$$

The Wronskian becomes

$$\varphi_1\varphi_2' - \varphi_1'\varphi_2 = e^2 - 1 \tag{61}$$

For elliptical orbits $0 \le e < 1$,

$$e^2 - 1 \neq 0 \tag{62}$$

therefore, the two solutions are linearly independent. When these linearly independent solutions are used, a particular solution of Eq. (38) can be obtained by the method of variation of parameters

$$\varphi_3 = \frac{2K_{x1}}{e^2 - 1}\left(\varphi_1\int\varphi_2\,\mathrm{d}\theta - \varphi_2\int\varphi_1\,\mathrm{d}\theta\right) \tag{63}$$

When

$$\int\varphi_1\,\mathrm{d}\theta = -\frac{1}{2e}\rho^2 \tag{64}$$

$$\int\varphi_2\,\mathrm{d}\theta = -\frac{3}{2}e\rho^2 J + \frac{1}{2}\sin\theta(\rho + 1) \tag{65}$$

are inserted, the particular solution becomes

$$\varphi_3 = -(K_{x1}/e)\rho\cos\theta \tag{66}$$

With appropriate integration constants K_{z1} and K_{z2}, the solution of Eq. (38) can now be written as

$$\tilde{z} = K_{z1}\varphi_1 + K_{z2}\varphi_2 + \varphi_3 = K_{z1}\rho\sin\theta + K_{z2}(3e^2\rho\sin\theta J + \rho\cos\theta - 2e) - (K_{x1}/e)\rho\cos\theta \tag{67}$$

This can be rearranged as

$$\tilde{z} = K_{z1}\rho\sin\theta + (K_{z2} - K_{x1}/e)\rho\cos\theta - K_{z2}e(2 - 3e\rho\sin\theta J) \tag{68}$$

We now find $\tilde{x}$ from Eq. (37) by substitution

$$\tilde{x}' = 2\tilde{z} + K_{x1} = 2K_{z1}\rho\sin\theta + 2(K_{z2} - K_{x1}/e)\rho\cos\theta - 2K_{z2}e(2 - 3e\rho\sin\theta J) + K_{x1} = 2K_{z1}\rho\sin\theta + (K_{z2} - K_{x1}/e)(2\rho\cos\theta - e) - 3K_{z2}e(1 - 2e\rho\sin\theta J) \tag{69}$$

It can be integrated as

$$\tilde{x} = K_{x2} - K_{z1}\cos\theta(\rho + 1) + (K_{z2} - K_{x1}/e)\sin\theta(\rho + 1) - 3K_{z2}e\rho^2 J \tag{70}$$

where K_{x2} is an integral constant.

For convenience, we redefine the integral constants as

$$K_1 \equiv K_{x2}, \qquad K_2 \equiv K_{z1}$$

$$K_3 \equiv [K_{z2} - (K_{x1}/e)], \qquad K_4 \equiv -K_{z2}e$$

after which Eqs. (68) and (70) become

$$\tilde{x} = K_1 - K_2\cos\theta(\rho + 1) + K_3\sin\theta(\rho + 1) + 3K_4\rho^2 J$$
$$\tilde{z} = K_2\rho\sin\theta + K_3\rho\cos\theta + K_4(2 - 3e\rho\sin\theta J) \tag{71}$$

If we use a notation similar to the one used for the CW solutions

$$s = \rho\sin\theta, \qquad c = \rho\cos\theta$$

the description of Eq. (71) becomes slightly simpler:

$$\tilde{x} = K_1 - K_2 c(1 + 1/\rho) + K_3 s(1 + 1/\rho) + 3K_4\rho^2 J$$
$$\tilde{z} = K_2 s + K_3 c + K_4(2 - 3esJ) \tag{72}$$

The equations for the velocity can be obtained by differentiation of those equations:

$$\tilde{v}_x = 2K_2 s + K_3(2c - e) + 3K_4(1 - 2esJ)$$
$$\tilde{v}_z = K_2 s' + K_3 c' - 3K_4 e(s'J + s/\rho^2) \tag{73}$$

Expressing the in-plane equations in a matrix notation we obtain

$$\begin{bmatrix}\tilde{x}\\ \tilde{z}\\ \tilde{v}_x\\ \tilde{v}_z\end{bmatrix} = \begin{bmatrix}1 & -c(1+1/\rho) & s(1+1/\rho) & 3\rho^2 J\\ 0 & s & c & (2-3esJ)\\ 0 & 2s & 2c-e & 3(1-2esJ)\\ 0 & s' & c' & -3e(s'J + s/\rho^2)\end{bmatrix} \times \begin{bmatrix}K_1\\ K_2\\ K_3\\ K_4\end{bmatrix} \equiv \Phi_\theta \boldsymbol{K} \tag{74}$$

where

$$s' = \cos\theta + e\cos 2\theta, \qquad c' = -(\sin\theta + e\sin 2\theta)$$
$$J = k^2(t - t_0), \qquad k^2 = h/p^2 \tag{75}$$

State Transition Matrix

What we finally need is the state transition matrix that can propagate the initial state to a final state for arbitrary time. This state transition matrix can be obtained as

$$\Phi_{\theta_0}^{\theta} = \Phi_\theta \Phi_{\theta_0}^{-1}$$

We eliminate the integration constants by inserting the initial conditions. We already have the solution matrix as Eq. (74):

$$\Phi_\theta = \begin{bmatrix}1 & -c(1+1/\rho) & s(1+1/\rho) & 3\rho^2 J\\ 0 & s & c & (2-3esJ)\\ 0 & 2s & 2c-e & 3(1-2esJ)\\ 0 & s' & c' & -3e(s'J + s/\rho^2)\end{bmatrix} \tag{76}$$

Furthermore, from Eq. (44), we obtain

$$J(\theta_0) = 0 \tag{77}$$

leading to

$$\Phi_{\theta_0} = \begin{bmatrix}1 & -c(1+1/\rho) & s(1+1/\rho) & 0\\ 0 & s & c & 2\\ 0 & 2s & 2c-e & 3\\ 0 & s' & c' & -3es/\rho^2\end{bmatrix} \tag{78}$$

To find the inverse matrix, we need the determinant of Φ_{θ_0}, which becomes

$$\det\Phi_{\theta_0} = \begin{vmatrix} s & c & 2\\ 2s & 2c-e & 3\\ s' & c' & -3es/\rho^2\end{vmatrix} = e^2 - 1 \neq 0 \quad \text{for } 0 \le e < 1 \tag{79}$$

It is, therefore, guaranteed that $\Phi_{\theta_0}^{-1}$ exists for all true anomalies and all eccentricities. The elements of $\Phi_{\theta_0}^{-1}$ become

$$\Phi_{\theta_0}^{-1} = \frac{1}{1-e^2} \times \begin{bmatrix}1-e^2 & 3e(s/\rho)(1+1/\rho) & -es(1+1/\rho) & -ec+2\\ 0 & -3(s/\rho)(1+e^2/\rho) & s(1+1/\rho) & c-2e\\ 0 & -3(c/\rho+e) & c(1+1/\rho)+e & -s\\ 0 & 3\rho+e^2-1 & -\rho^2 & es\end{bmatrix}_{\theta_0} \tag{80}$$

Solution Algorithm

Given the initial relative position and velocity in transformed coordinates as

$$\tilde{\boldsymbol{r}}_0 = \begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \tilde{z}_0\end{bmatrix}, \qquad \tilde{\boldsymbol{v}}_0 = \begin{bmatrix}\tilde{v}_{x0}\\ \tilde{v}_{y0}\\ \tilde{v}_{z0}\end{bmatrix} \tag{81}$$

and the initial value of the true anomaly of the target spacecraft as θ_0, one obtains the transition matrix for in-plane and out-of-plane motions, respectively.

In-Plane

The pseudoinitial values can be calculated from the initial values as

$$\begin{bmatrix}\bar{x}_0\\ \bar{z}_0\\ \cdots\\ \bar{v}_{x0}\\ \bar{v}_{z0}\end{bmatrix} = \frac{1}{1-e^2}\left[\begin{array}{cc:cc}1-e^2 & 3es(1/\rho+1/\rho^2) & -es(1+1/\rho) & -ec+2\\ 0 & -3s(1/\rho+e^2/\rho^2) & s(1+1/\rho) & c-2e\\ \hdashline 0 & -3(c/\rho+e) & c(1+1/\rho)+e & -s\\ 0 & 3\rho+e^2-1 & -\rho^2 & es\end{array}\right]_{\theta_0} \begin{bmatrix}\tilde{x}_0\\ \tilde{z}_0\\ \cdots\\ \tilde{v}_{x0}\\ \tilde{v}_{z0}\end{bmatrix} \tag{82}$$

Once these pseudoinitial values have been obtained, the relative state at arbitrary time can be calculated as follows:

$$\begin{bmatrix}\tilde{x}_t\\ \tilde{z}_t\\ \cdots\\ \tilde{v}_{xt}\\ \tilde{v}_{zt}\end{bmatrix} = \left[\begin{array}{cc:cc}1 & -c(1+1/\rho) & s(1+1/\rho) & 3\rho^2 J\\ 0 & s & c & (2-3esJ)\\ \hdashline 0 & 2s & 2c-e & 3(1-2esJ)\\ 0 & s' & c' & -3e(s'J+s/\rho^2)\end{array}\right]_{\theta} \begin{bmatrix}\bar{x}_0\\ \bar{z}_0\\ \cdots\\ \bar{v}_{x0}\\ \bar{v}_{z0}\end{bmatrix} \tag{83}$$

where θ can be calculated at any time using Kepler's equation.

Out-of-Plane

The relative state for the out-of-plane at arbitrary time can be calculated by

$$\begin{bmatrix}\tilde{y}_t\\ \tilde{v}_{yt}\end{bmatrix} = \frac{1}{\rho_{\theta-\theta_0}}\begin{bmatrix}c & s\\ -s & c\end{bmatrix}_{\theta-\theta_0}\begin{bmatrix}\tilde{y}_0\\ \tilde{v}_{y0}\end{bmatrix} \tag{84}$$

where

$$\rho = 1 + e\cos\theta, \qquad s = \rho\sin\theta, \qquad c = \rho\cos\theta$$
$$s' = \cos\theta + e\cos 2\theta, \qquad c' = -(\sin\theta + e\sin 2\theta)$$
$$J = k^2(t - t_0), \qquad k^2 = h/p^2 \tag{85}$$

Table 1 Simulation conditions

Parameter	Value
Target orbit	
Eccentricity	$e = 0.1$ and 0.7
Perigee height	500 km
Inclination	30 deg
Longitude of the ascending node	0 deg
Argument of perigee	0 deg
Initial true anomaly	45 deg
Chaser initial position in Hill coordinate frame	[100,10,10] m
Chaser initial velocity in Hill coordinate frame	[0.1,0.1,0.1] m/s
Propagate duration	Two-orbit revolutions
Numerical integration method	Runge–Kutta method Fourth order Variable step size

Recall that the transformed variables are calculated from the true values using

$$\tilde{\boldsymbol{r}} = \rho \boldsymbol{r}, \qquad \tilde{\boldsymbol{v}} = -e\sin\theta \boldsymbol{r} + (1/k^2\rho)\boldsymbol{v} \tag{86}$$

The inverse transformation is

$$\boldsymbol{r} = 1/\rho\tilde{\boldsymbol{r}}, \qquad \boldsymbol{v} = k^2(e\sin\theta\tilde{\boldsymbol{r}} + \rho\tilde{\boldsymbol{v}}) \tag{87}$$

Relationship to CW Solutions

So far, we have developed the general case for arbitrary elliptic orbits. We will check the relation between the new method of Eqs. (82–84) and the conventional CW solutions. Setting $e = 0$ in Eq. (85), we obtain:

$$\rho = 1, \qquad s = \sin\theta, \qquad c = \cos\theta$$

$$J = k^2(t - t_0) = \omega(t - t_0) \tag{88}$$

This leads to the following transition matrix for in-plane motion:

$$\begin{bmatrix} \tilde{x}_t \\ \tilde{z}_t \\ \cdots \\ \tilde{v}_{xt} \\ \tilde{v}_{zt} \end{bmatrix} = \left[\begin{array}{cc:cc} 1 & -2\cos\theta & 2\sin\theta & 3\omega(t-t_0) \\ 0 & \sin\theta & \cos\theta & 2 \\ \hdashline 0 & 2\sin\theta & 2\cos\theta & 3 \\ 0 & \cos\theta & -\sin\theta & 0 \end{array}\right] \left[\begin{array}{cc:cc} 1 & 0 & 0 & 2 \\ 0 & -3\sin\theta_0 & 2\sin\theta_0 & \cos\theta_0 \\ \hdashline 0 & -3\cos\theta_0 & 2\cos\theta_0 & -\sin\theta_0 \\ 0 & 2 & -1 & 0 \end{array}\right] \begin{bmatrix} \tilde{x}_0 \\ \tilde{z}_0 \\ \cdots \\ \tilde{v}_{x0} \\ \tilde{v}_{z0} \end{bmatrix}$$

$$= \left[\begin{array}{cc:cc} 1 & 6(\Delta\theta - \sin\Delta\theta) & 4\sin\Delta\theta - 3\Delta\theta & 2(1-\cos\Delta\theta) \\ 0 & 4 - 3\cos\Delta\theta & 2(\cos\Delta\theta - 1) & \sin\Delta\theta \\ \hdashline 0 & 6(1-\cos\Delta\theta) & 4\cos\Delta\theta - 3 & 2\sin\Delta\theta \\ 0 & 3\sin\Delta\theta & -2\sin\Delta\theta & \cos\Delta\theta \end{array}\right] \begin{bmatrix} \tilde{x}_0 \\ \tilde{z}_0 \\ \cdots \\ \tilde{v}_{x0} \\ \tilde{v}_{z0} \end{bmatrix}, \qquad \Delta t = t - t_0, \qquad \Delta\theta = \theta - \theta_0 \tag{89}$$

From Eq. (86), we get for $e = 0$

$$\tilde{\boldsymbol{r}} = \boldsymbol{r}, \qquad \tilde{\boldsymbol{v}} = (1/\omega)\boldsymbol{v} \tag{90}$$

consequently,

$$\begin{bmatrix} x_t \\ z_t \\ \cdots \\ v_{xt} \\ v_{zt} \end{bmatrix} = \left[\begin{array}{cc:cc} 1 & 6(\Delta\theta - \sin\Delta\theta) & (1/\omega)(4\sin\Delta\theta - 3\Delta\theta) & (2/\omega)(1-\cos\Delta\theta) \\ 0 & 4 - 3\cos\Delta\theta & (2/\omega)(\cos\Delta\theta - 1) & \sin\Delta\theta/\omega \\ \hdashline 0 & 6\omega(1-\cos\Delta\theta) & 4\cos\Delta\theta - 3 & 2\sin\Delta\theta \\ 0 & 3\omega\sin\Delta\theta & -2\sin\Delta\theta & \cos\Delta\theta \end{array}\right] \begin{bmatrix} x_0 \\ z_0 \\ \cdots \\ v_{x0} \\ v_{z0} \end{bmatrix} \tag{91}$$

This is exactly the CW equations, and, therefore, Eqs. (82) and (83) include the CW equations as a special case for $e = 0$. Similarly, Eq. (84) for out-of-plane motion becomes

$$\begin{bmatrix} y_t \\ v_{yt} \end{bmatrix} = \begin{bmatrix} \cos\Delta\theta & (1/\omega)\sin\Delta\theta \\ -\omega\sin\Delta\theta & \cos\Delta\theta \end{bmatrix} \begin{bmatrix} y_0 \\ v_{y0} \end{bmatrix} \tag{92}$$

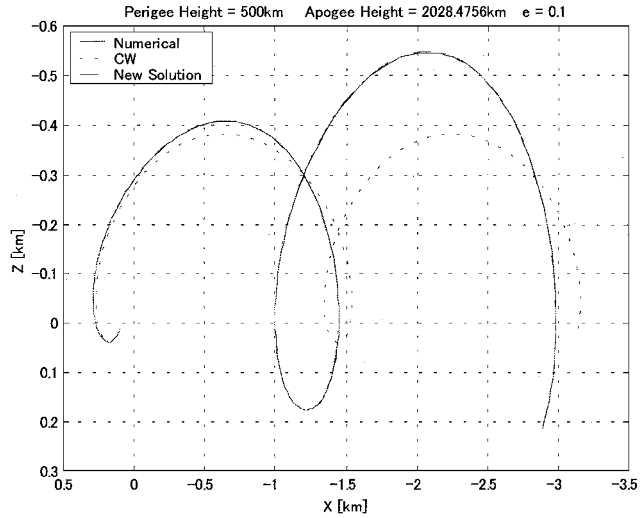

Fig. 2 X–Z plane, $e = 0.1$.

This is also exactly the CW equation for out-of-plane motion. Equations (82–84) are applicable to all elliptical orbits including circular orbit.

Numerical Verification

Because Eqs. (82–84) are analytically exact solutions of the equations of the linear motion (15), they must be numerically exact if the equation of motion (15) is valid, that is, if

$$|\boldsymbol{R}| \gg |\boldsymbol{r}|$$

holds. To verify the exactness of Eqs. (82–84), we compare with the numerical solutions of the nonlinear equations. To obtain numerical solutions, we integrated the equations of orbit motion of chaser and target, respectively, in an Earth-centered inertial frame using a fourth-order Runge–Kutta method with variable step size. Then the chaser state vector is subtracted from the target state vector and converted into Hill's rotating frame. These numerical values are compared with the results of Eqs. (82–84). Simulation conditions are summarized in Table 1.

We show two cases, $e = 0.1$ and 0.7 (see Figs. 2–5). It can be seen from the results that Eqs. (82–84) are accurate. It can also be seen that the CW solutions cause significant errors in the elliptical RV case.

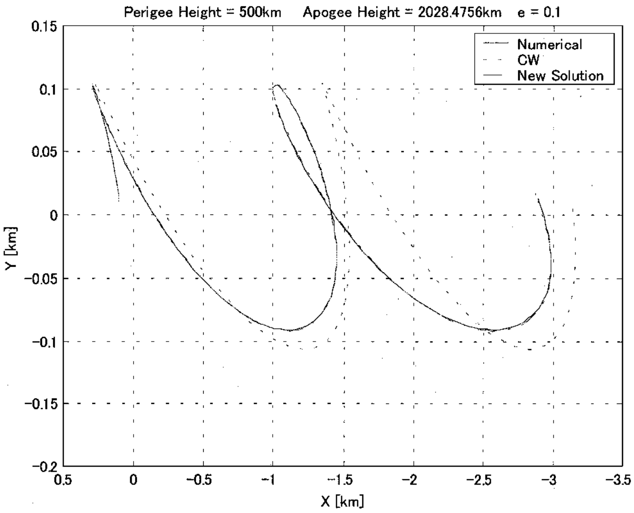

Fig. 3 *X–Y* plane, $e = 0.1$.

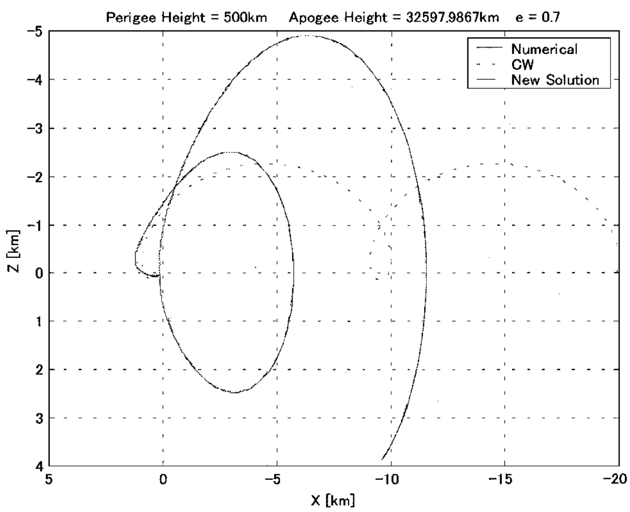

Fig. 4 *X–Z* plane, $e = 0.7$.

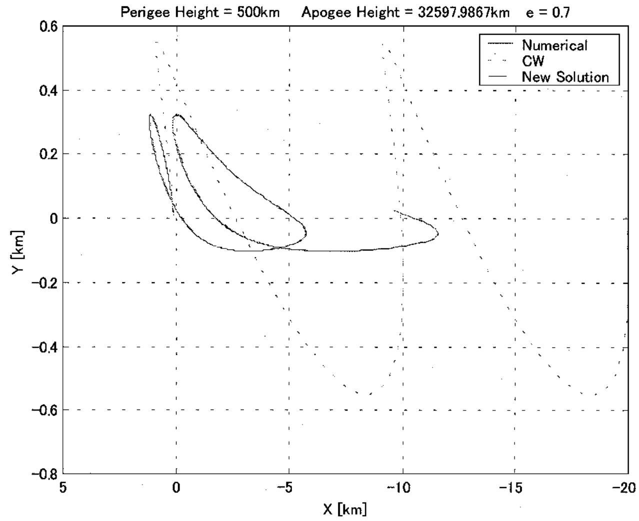

Fig. 5 *X–Y* plane, $e = 0.7$.

Conclusions

A simpler form of the state transition matrix has been developed for the relative motion between two spacecraft on an elliptical orbit. This solution, which is the most promising for practical engineering use, has been achieved by using a simpler integral function related to the orbital dynamics for finding the solution to the differential equations of relative motion. This new state transition matrix is valid for arbitrary elliptical orbits of $0 \leq e < 1$. Given the initial relative position and velocity of the chaser spacecraft and the initial value of the true anomaly of the target spacecraft, the relative state at arbitrary time can be calculated for an arbitrary elliptical orbit without numerical integration. This method has been verified by extensive numerical simulations and shows good agreement with numerical results.

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